Monday, November 9, 2015

p. 549: 43, 44, 45, 47, 49, 52, 55, 88, 99, 100

Problem 43

Problem. Evaluate the limit $\lim_{x\to\infty} x\ln x$ using L'Hôpital's Rule if necessary.

The form is $\infty \cdot \infty$, which is not indeterminate. The limit is ∞.

Solution.

Problem 44

Problem. Evaluate the limit $\lim_{x\to 0^+} x^3 \cot x$ using L'Hôpital's Rule if necessary.

Solution. The form is $0 \cdot \infty$, which is indeterminate.

$$\begin{aligned} \lim_{x\to 0^+} x^3 \cot x &= \lim_{x\to 0^+} \frac{x^3}{\tan x} \\ &= \lim_{x\to 0^+} \frac{3x^2}{\sec^2 x} \\ &= 0. \end{aligned}$$

Problem 45

Problem. Evaluate the limit $\lim_{x\to\infty} x \sin \dfrac{1}{x}$ using L'Hôpital's Rule if necessary.

Solution. The form is $\infty \cdot 0$, which is indeterminate.

$$\begin{aligned} \lim_{x\to\infty} x\sin\frac{1}{x} &= \lim_{x\to\infty} \frac{\sin\frac{1}{x}}{1/x} \\ &= \lim_{x\to\infty} \frac{\cos\frac{1}{x}\cdot\left(-\frac{1}{x^2}\right)}{-1/x^2} \\ &= \lim_{x\to\infty} \cos\frac{1}{x} \\ &= \cos 0 \\ &= 1. \end{aligned}$$

Problem 47

Problem. Evaluate the limit $\lim_{x\to 0^+} x^{1/x}$ using L'Hôpital's Rule if necessary.

Solution. The form is 0^∞, which is not indeterminate. The limit is 0.

Problem 49

Problem. Evaluate the limit $\lim_{x\to\infty} x^{1/x}$ using L'Hôpital's Rule if necessary.

Solution. The limit is ∞^0, which is indeterminate.

$$\begin{aligned}\ln \lim_{x\to\infty} x^{1/x} &= \lim_{x\to\infty} \ln x^{1/x} \\ &= \lim_{x\to\infty} \frac{1}{x} \cdot \ln x \\ &= \lim_{x\to\infty} \frac{\ln x}{x} \\ &= \lim_{x\to\infty} \frac{1/x}{1} \\ &= 0.\end{aligned}$$

Therefore, $\lim_{x\to\infty} x^{1/x} = e^0 = 1$.

Problem 52

Problem. Evaluate the limit $\lim_{x\to\infty} (1+x)^{1/x}$ using L'Hôpital's Rule if necessary.

Solution. The form is ∞^0, which is indeterminate.

$$\begin{aligned}\ln \lim_{x\to\infty} (1+x)^{1/x} &= \lim_{x\to\infty} \ln (1+x)^{1/x} \\ &= \lim_{x\to\infty} \frac{1}{x} \cdot \ln (1+x) \\ &= \lim_{x\to\infty} \frac{\ln (1+x)}{x} \\ &= \lim_{x\to\infty} \frac{1/(1+x)}{1} \\ &= 0.\end{aligned}$$

Therefore, $\lim_{x\to\infty} (1+x)^{1/x} = e^0 = 1$.

Problem 55

Problem. Evaluate the limit $\lim_{x\to 1^+} (\ln x)^{x-1}$ using L'Hôpital's Rule if necessary.

Solution. The form is 0^0, which is indeterminate.

$$\begin{aligned}
\ln \lim_{x\to 1^+} (\ln x)^{x-1} &= \lim_{x\to 1^+} \ln (\ln x)^{x-1} \\
&= \lim_{x\to 1^+} (x-1)\ln\ln x \\
&= \lim_{x\to 1^+} \frac{\ln\ln x}{1/(x-1)} \\
&= \lim_{x\to 1^+} \frac{\frac{1}{\ln x}\cdot\frac{1}{x}}{-1/(x-1)^2} \\
&= -\lim_{x\to 1^+} \frac{(x-1)^2}{x\ln x} \\
&= -\lim_{x\to 1^+} \frac{2(x-1)}{x\cdot\frac{1}{x}+\ln x} \\
&= -\lim_{x\to 1^+} \frac{2(x-1)}{1+\ln x} \\
&= 0.
\end{aligned}$$

Therefore, $\lim_{x\to 1^+} (\ln x)^{x-1} = e^0 = 1$.

Problem 88

Problem. The formula for the amount A in a savings account compounded n times per year for t years at an interest rate r and an initial deposit of P is given by

$$A = P\left(1+\frac{r}{n}\right)^{nt}.$$

Use L'Hôpital's Rule to show that the limiting formula as the number of compoundings per year approaches infinity is given by $A = Pe^{rt}$.

Solution.

$$\begin{aligned}
\ln \lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} &= \lim_{n\to\infty} \ln P\left(1+\frac{r}{n}\right)^{nt} \\
&= \ln P + \lim_{n\to\infty} \ln\left(1+\frac{r}{n}\right)^{nt} \\
&= \ln P + \lim_{n\to\infty} nt\ln\left(1+\frac{r}{n}\right) \\
&= \ln P + \lim_{n\to\infty} \frac{\ln P\left(1+\frac{r}{n}\right)}{1/(nt)} \\
&= \ln P + \lim_{n\to\infty} \frac{\ln P + \ln\left(1+\frac{r}{n}\right)}{1/(nt)} \\
&= \ln P + \lim_{n\to\infty} \frac{\left(\frac{1}{1+\frac{r}{n}}\right)\cdot\left(-\frac{r}{n^2}\right)}{\left(-\frac{1}{n^2 t}\right)} \\
&= \ln P + \lim_{n\to\infty} \frac{rt}{1+\frac{r}{n}} \\
&= \ln P + rt.
\end{aligned}$$

Therefore, $\lim_{n\to\infty} P\left(1+\frac{r}{n}\right)^{nt} = e^{\ln P + rt} = Pe^{rt}$.

Problem 99

Problem. Find the limit, as x approaches 0, of the ratio of the area of the triangle to the total shaded are in the figure (shown in the book).

Solution.

Problem 100

Problem. In Section 1.3, a geometric argument was used to prove that

$$\lim_{\theta\to 0} \frac{\sin\theta}{\theta} = 1.$$

(a) Write the area of $\triangle ABD$ in terms of θ (See figure in the book).

(b) Write the area of the shaded region in terms of θ.

(c) Write the ratio R of the area of $\triangle ABD$ to that of the shaded region.

(d) Find $\lim_{\theta\to 0} R$.

Solution.